Equations of Stellar Structure

Stellar structure and evolution can be calculated via a series of differential equations involving mass, pressure, temperature, and density. For simplicity, we will assume spherical symmetry.
First, let's define the global variables

$$
\begin{aligned}
r &= \text{the radial coordinate of the star} \\
\mathcal{M}(r) &= \text{the stellar mass contained within radius, } r \\
\mathcal{L}(r) &= \text{the luminosity generated within radius } r \\
P(r) &= \text{ion + electron + radiation pressure at } r \\
\rho(r) &= \text{the density at } r \\
X(r) &= \text{the mass fraction of the star that is hydrogen} \\
Y(r) &= \text{the mass fraction of the star that is helium} \\
Z(r) &= \text{the mass fraction of the star that is in metals} \\
x_i(r) &= \text{the mass fraction of the star in species } i \\
\mu(r) &= \text{the mean molecular weight at } r \\
\mathcal{M}_T &= \text{Total Mass of Star} \\
\mathcal{L}_T &= \text{Total Luminosity of Star} \\
R &= \text{the Radius of the star} \\
t &= \text{the independent time variable}
\end{aligned}
$$

Obviously

$$X + Y + Z = 1$$

Note: I've written the above parameters in terms the radial coordinate of the star. However, for many reasons, it is better to use the Lagrangian formulation, and use $\mathcal{M}$, the enclosed mass, and the independent variable. The dependent variables are then $r(\mathcal{M})$, $\mathcal{L}(\mathcal{M})$, $P(\mathcal{M})$, etc.

In addition, we will also define some local variables:

$$
\begin{aligned}
\epsilon &= \quad \text{the luminosity produced per unit mass of material} \\
\kappa_\nu &= \quad \text{the opacity (photon cross section) per unit mass of material.}
\end{aligned}
$$

Conservation of Mass

For a spherically symmetric star, the mass interior to some radius R is

$$\mathcal{M}(r) = \int_0^R 4\pi r^2 \rho(r)\, dr \tag{2.1.1}$$

Written in terms of the differential, this is

$$\frac{d\mathcal{M}}{dr} = 4\pi r^2 \rho(r) \tag{2.1.2}$$

However, over its lifetime, a star's radius will change by many orders of magnitude, while its mass will remain relatively constant. Moreover, the amount of nuclear reactions occuring inside a star depends on the density and temperature, not where it is in the star. Thus, as stated above, a better and more natural way to treat stellar structure is to write radius as a function of mass, i.e.,

$$\frac{dr}{d\mathcal{M}} = \frac{1}{4\pi r^2 \rho} \tag{2.1.3}$$

This is the Lagrangian form of the equation (rather than the Eulerian form). All the equations of stellar structure will be expressed in the Lagrangian form, and **most of the parameters** will be expressed in **per unit mass**, rather than per unit size or volume.

Conservation of Momentum

If a star is in Hydrostatic Equilibrium, then the force of gravity at any point must be balanced by the pressure. Consider a shell at radius r: the mass per unit area in the shell is $\rho\, dr$, and the weight (per unit area) is $-g\rho\, dr$. This weight must cancel the pressure difference felt from one side of the shell to the other, $P_{\text{inner}} - P_{\text{outer}} = \frac{dP}{dr} dr$. Thus

$$\frac{dP}{dr} = -g\rho \tag{2.2.1}$$

Since the force of gravity, g is just $G\mathcal{M}/r^2$, then

$$\frac{dP}{dr} = -\frac{G\mathcal{M}(r)}{r^2}\rho \tag{2.2.2}$$

So if we multiply by the conservation of mass equation, we get the Lagrangian form of hydrostatic equilibrium

$$\frac{dP}{dr}\frac{dr}{d\mathcal{M}} = -\left(\frac{G\mathcal{M}}{r^2}\rho\right)\left(\frac{1}{4\pi r^2 \rho}\right) \tag{2.2.3}$$

or

$$\frac{dP}{d\mathcal{M}} = -\frac{G\mathcal{M}}{4\pi r^4} \tag{2.2.4}$$

Note, however, that stars need not be in hydrostatic equilibrium. Some stars pulsate! In these cases, the pressure at any moment may not equal the force of gravity, causing the shell to undergo an acceleration.

Again, consider a shell at radius r. The force per unit area acting upon this shell is simply the difference in pressure from one side of the shell to the other

$$f_P = -\frac{dP}{dr}dr \tag{2.2.5}$$

while the force per unit area due to gravity is

$$f_g = g\,\rho\,dr = \frac{G\mathcal{M}(r)}{r^2}\rho\,dr \tag{2.2.6}$$

These two forces define the acceleration of the shell

$$f_P + f_g = -\left(\frac{dP}{dr}dr\right) + \left(\frac{G\mathcal{M}}{r^2}\rho\,dr\right) = \rho\,dr\left(\frac{d^2r}{dt^2}\right) \tag{2.2.7}$$

Again, this should be translated to the Lagrangian form by substituting $d\mathcal{M}$ for dr using the mass conservation equation

$$f_P + f_g = -\left(\frac{dP}{d\mathcal{M}}d\mathcal{M}\right) + \left(\frac{G\mathcal{M}}{4\pi r^4}d\mathcal{M}\right) = \left(\frac{d\mathcal{M}}{4\pi r^2}\right)\left(\frac{d^2r}{dt^2}\right) \tag{2.2.8}$$

or

$$\left(\frac{dP}{d\mathcal{M}}\right) = -\left(\frac{G\mathcal{M}}{4\pi r^4}\right) - \left(\frac{1}{4\pi r^2}\right)\left(\frac{d^2r}{dt^2}\right) \tag{2.2.9}$$

Note that if the acceleration is zero, the equation reduces to the simple hydrostatic equation.

Conservation of Energy

Consider the net energy per second passing outward through a shell at radius r. If no energy is created in the shell, then the amount of energy in equals the amount of energy out, and $d\mathcal{L}/dr = 0$. However, if additional energy is created or absorbed within the shell, then $d\mathcal{L}/dr$ will be non-zero. Let's define ϵ as the energy released per second by a unit mass of matter. Then

$$\frac{d\mathcal{L}}{dr} = 4\pi r^2 \rho\, \epsilon \tag{2.3.1}$$

or, in the Lagrangian form

$$\frac{d\mathcal{L}}{d\mathcal{M}} = \epsilon \tag{2.3.2}$$

Note that ϵ has three components.

1) ϵ_n, the total energy created by nuclear reactions

2) ϵ_ν, the energy input into neutrinos, and

3) ϵ_g, the energy produced or lost by gravitational expansion or contraction.

Thus,

$$\frac{d\mathcal{L}}{d\mathcal{M}} = \epsilon_n - \epsilon_\nu + \epsilon_g \tag{2.3.3}$$

In general, the contribution from nuclear reactions will always be positive, while the energy in neutrinos will always be lost from the system. (That's why there's a negative sign before ϵ_ν. But note that supernovae are the exception to this rule.) The gravitational energy can be either positive or negative, and simply reflects the heat gained or lost in the shell by PdV work. Thus,

$$\epsilon_g = -\frac{dq}{dt}$$

which, from thermodynamics (1.18), is

$$\epsilon_g = -\frac{dq}{dt} = -c_P \frac{dT}{dt} + \frac{\delta}{\rho}\frac{dP}{dt} \tag{2.3.4}$$

This can sometimes be written in a more useful form by recalling that

$$\nabla_{\mathrm{ad}} = \frac{P\delta}{T\rho c_P} \tag{1.19}$$

Thus

$$\epsilon_g = -c_P T \left(\frac{1}{T}\frac{dT}{dt} - \frac{\nabla_{\mathrm{ad}}}{P}\frac{dP}{dt} \right) \tag{2.3.5}$$

Thermal Structure

The thermal structure of the star will depend on its method for transporting energy from its interior to its surface. For convenience, we will express the temperature stratification of the star in terms of the temperature gradient

$$\nabla = \left(\frac{d \ln T}{d \ln P} \right) \tag{2.4.1}$$

This temperature structure is then simply

$$\frac{dT}{dr} = \frac{dP}{dr} \frac{dT}{dP} = \frac{dP}{dr} \frac{T}{P} \nabla \tag{2.4.2}$$

If the star is in approximate hydrostatic equilibrium, this simplifies to

$$\frac{dT}{dr} = -\frac{G \mathcal{M} \rho T}{r^2 P} \nabla \tag{2.4.3}$$

or, in Lagrangian terms

$$\frac{dT}{d\mathcal{M}} = -\frac{G \mathcal{M} T}{4 \pi r^4 P} \nabla \tag{2.4.4}$$